THE STRUCTURE OF CROSSED PRODUCTS BY ENDOMORPHISMS

EDUARD ORTEGA AND ENRIQUE PARDO

Abstract. We describe simplicity of the Stacey crossed product $A \times_\beta \mathbb{N}$ in terms of conditions of the endomorphism β. Then, we use a characterization of the graph C^*-algebras $C^*(E)$ as the Stacey crossed product $C^*(E)^\gamma \times_{\beta_E} \mathbb{N}$ to study its ideal properties, in terms of the (non-classical) C^*-dynamical system $(C^*(E)^\gamma, \beta_E)$. Finally, we give sufficient conditions for the Stacey crossed product $A \times_\beta \mathbb{N}$ being a purely infinite simple C^*-algebra.

In [12], Cuntz defined the fundamental Cuntz algebras $\mathcal{O}_n$. He also represented these algebras as crossed products of a UHF-algebra by an endomorphism, and he used this representation to prove the simplicity of his algebras. In a subsequent paper [13] he saw this construction as a full corner of an ordinary crossed product. However Cuntz did not explain what kind of crossed product by an endomorphism was. Later, Paschke [26] gave an elegant generalization of Cuntz's result, and described the crossed product of a unital C^*-algebra by an endomorphism $\beta : A \to A$, written $A \times_\beta \mathbb{N}$, as the C^*-algebra generated by A and an isometry V, such that $VaV^* = \beta(a)$. Endomorphisms of C^*-algebras appeared elsewhere (cf. [7], [14] and the references given there), and this led Stacey to give a modern description of their crossed products in terms of covariant representations and universal properties [33]. He also verified that the candidate proposed in [12] had the required property. See [3] and [11] for further study and generalization of the Stacey's crossed product.

Cuntz's representation of the $\mathcal{O}_n$ as crossed products by an endomorphism aimed to prove the simplicity of these C^*-algebras. Paschke gave conditions on the C^*-algebra A and in the isometry to obtain a simple crossed product [26, Proposition 2.1], later improved in [11, Corollary 2.6]. But it is in [32, Theorem 4.1] where the most powerful result about the simplicity of the Stacey crossed product is given. Namely, If A is a unital C^*-algebra and β is an injective $*$-endomorphism, then $A \times_\beta \mathbb{N}$ is simple and $\beta(1)$ is a full projection in A if and only if β^n is outer for every $n > 0$ and there are no non-trivial ideals I of A with $\beta(I) \subseteq I$. Schweizer used the representation of the Stacey crossed product as Cuntz Pimsner algebra given by Muhly and Solel [21].

The theory of graph C^*-algebras $C^*(E)$ has been developed by a number of researchers (see [8], [9] and [28], among others) in an attempt to produce a far-reaching and yet accessible

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generalization of the Cuntz-Krieger algebras of finite matrices. Indeed, graph algebras do provide a large and interesting class of examples of C^*-algebras, both simple and non-simple ones. For example, Cuntz's algebras are C^*-algebras of a graph.

In [4] an Huef and Raeburn study the crossed products of an Exel system, and they prove that the relative Cuntz-Pimsner algebra of an Exel system is isomorphic to a Stacey crossed product of its core. This result leads them to a realization of the graph algebra $C^*(E)$ as a Stacey crossed product $C^*(E)^\gamma \times_{\beta_E} \mathbb{N}$ by an endomorphism of the core, extending the work of Kwaśniewski on finite graphs [20].

In the case of Leavitt path algebras (see e.g. [2]), this result appears in more simple form in [6, Section 2], where the authors give a representation of Leavitt path algebras of a finite graph without sinks and sources as a fractional skew monoid rings (the algebraic analog of the crossed product by an endomorphism).

The aim of this paper is to study the simplicity of the non-unital crossed product. Our fundamental technique is seeing the Stacey crossed product $A\times_\beta \mathbb{N}$ as a full corner of a crossed product by an automorphism $P(A_\infty \times_{\beta_\infty} \mathbb{Z})P$ (see [13, 33]), where P is a full projection of the multipliers that is invariant under the canonical gauge-action. Therefore, we can define the associated *Connes' Spectrum* of the endomorphism in a similar way we do it for an automorphism (see [22, 23, 16]) and construct a parallel Connes' spectrum theory for endomorphisms. Hence, following the results of Olesen and Pedersen [23, 25] we characterize simplicity for the Stacey crossed product $A \times_\beta \mathbb{N}$.

As an example, we use the characterization of graph C^*-algebras $C^*(E)$ as Stacey crossed product $C^*(E)^\gamma \times_{\beta_E} \mathbb{N}$ [4], where in this case $C^*(E)^\gamma$ the core, that is a (non-unital) AF-algebra, and β_E is a corner isomorphism. However, although the characterization of the simplicity of $C^*(E)$ is well understood in terms of properties of the graph [8], our intention is to describe this characterization in terms of the non-classical C^*-dynamical system $(C^*(E)^\gamma, \beta_E)$. We also give conditions on the C^*-dynamical system to satisfy the Cuntz-Krieger uniqueness theorem: for any faithful covariant representation (π, V) of $(C^*(E)^\gamma, \beta_E)$ we have $C^*(\pi, V) \cong C^*(E)$. Finally, by using ideas from [29, 16], we give sufficient conditions on A and the endomorphism β in order to guarantee that $A \times_\beta \mathbb{N}$ is simple and purely infinite. The main difference between these previous results and ours is that we do not ask the C^*-algebra A to be simple.

The contents of this paper can be summarized as follows: In Section 1 we give the basic definitions of a Stacey crossed product. We use the characterization of the Stacey crossed product as a Cuntz Pimsner algebra [21] to describe the gauge invariant ideals, using a result from Katsura [18]. Then we define the Connes' spectrum of an endomorphism [22], a technical device that, with the help of results from Olsen and Pedersen ([23, 25]), allows us to give necessary and sufficient conditions to state the simplicity of a Stacey crossed product. In Section 2 we apply our results to graph C^*-algebras. We recall the definition of the graph endomorphism $\beta_E : C^*(E)^\gamma \to C^*(E)^\gamma$ of the core of the graph C^*-algebra [4, Theorem 9.3], used to prove that $C^*(E) \cong C^*(E)^\gamma \times_{\beta_E} \mathbb{N}$. Then, we characterize condition (L) of the graph E in terms of the endomorphism β_E: every cycle has an entry. Condition (L) in E implies that $C^*(E)$ satisfies the Cuntz-Krieger uniqueness theorem (see e.g. [28, Section 2]). Thus, we use our previous results to give the (well-known) necessary and sufficient condition of the graph E for the graph C^*-algebra $C^*(E)$ being simple. Finally, in Section 3, we give sufficient

conditions on the C^*-algebra A and the endomorphism $\beta : A \longrightarrow A$ for $A \times_\beta \mathbb{N}$ being a unital simple and purely infinite C^*-algebra.

1. Simple Stacey crossed product

The pair (A, β), where A is a C^*-algebra and $\beta : A \to A$ an injective endomorphism, is called a *C^*-dynamical system.*

Definition 1.1. We say that (π, V) is a *Stacey covariant representation* of (A, β) if $\pi : A \to \mathcal{B}(\mathcal{H})$ is a non-degenerated representation and V is an isometry of $\mathcal{B}(\mathcal{H})$ such that $\pi(\beta(a)) = V\pi(a)V^*$ for every $a \in A$. We say that (π, V) is *faithful* if π is faithful, and we denote by $C^*(\pi, V)$ the C^*-algebra generated by $\{\pi(A)V^n(V^m)^*\}_{n,m\geq 0}$.

Stacey showed in [33] that there exists a C^*-algebra that is generated by a universal Stacey covariant representation (ι_∞, V_∞). We call $A \times_\beta \mathbb{N} := C^*(\iota_\infty, V_\infty)$ the *Stacey crossed product* of A by the endomorphism β.

Remark 1.2. Observe that, if β is an automorphism, then $A \times_\beta \mathbb{N}$ is the usual crossed product $A \times_\beta \mathbb{Z}$.

Given $z \in \mathbb{T}$, we define an automorphism in $A\times_\beta\mathbb{N}$ by the rule $\gamma_z(a) = a$ and $\gamma_z(V_\infty) = zV_\infty$ for every $a \in A$. It defines the gauge action $\gamma : \mathbb{T} \to \mathrm{Aut}(A \times_\beta \mathbb{N})$. An ideal I of $A \times_\beta \mathbb{N}$ is said to be *gauge invariant* if $\gamma_z(I) = I$ for every $z \in \mathbb{T}$. We define a canonical faithful conditional expectation $E : A \times_\beta \mathbb{N} \longrightarrow A$ as $E(x) := \int_{\mathbb{T}} \gamma_z(x)dz$ for every $x \in A \times_\beta \mathbb{N}$.

We say that the endomorphism $\beta : A \longrightarrow A$ is *extendible* if, given any strong convergent sequence $\{x_n\}_{n\geq 0} \subset A$, then the sequence $\{\beta(x_n)\}_{n\geq 0}$ converges in the strong topology (i.e., β extends to $\widehat{\beta} : M(A) \longrightarrow M(A)$). Observe that, if β is injective, then $\widehat{\beta}(a) \in A$ implies that $a \in A$. Indeed, let $\{a_n\}$ be a sequence that converges in the strong topology and such that $\{\beta(a_n)\}$ converges in norm topology. Since β is isometric (β is injective) then $\{a_n\}$ converges in the norm topology too.

We define the inductive system $\{A_i, \gamma_i\}_{i\geq 0}$ given by $A_i := A$ and $\gamma_i = \beta$ for every $i \geq 0$. Let $A_\infty := \varinjlim\{A_i, \gamma_i\}$. For any $i \geq 0$, $\varphi_i : A_i \longrightarrow A_\infty$ denotes the (injective) canonical map. The diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\beta} & A & \xrightarrow{\beta} & A & \xrightarrow{\beta} & \cdots \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\beta} & & \\ A & \xrightarrow[\beta]{} & A & \xrightarrow[\beta]{} & A & \xrightarrow[\beta]{} & \cdots \end{array}$$

gives rise to an automorphism $\beta_\infty : A_\infty \longrightarrow A_\infty$.

Observe that, if β is an extendible endomorphism, then φ_0 extends to $\widehat{\varphi_0} : M(A) \longrightarrow M(A_\infty)$.

Proposition 1.3 (cf. [32, Proposition 3.3])**.** *If A is a C^*-algebra and $\beta : A \longrightarrow A$ is an extendible and injective endomorphism, then $A \times_\beta \mathbb{N} \cong P(A_\infty \times_{\beta_\infty} \mathbb{Z})P$, where $P = \widehat{\varphi_0}(1_{M(A)}) \in M(A_\infty \times_{\beta_\infty} \mathbb{Z})$. Moreover, P is a full projection, so that $A \times_\beta \mathbb{N}$ is strongly Morita equivalent to $A_\infty \times_{\beta_\infty} \mathbb{Z}$.*

Therefore, there there exist a bijection between the ideals of $A_\infty \times_{\beta_\infty} \mathbb{Z}$ and $A \times_\beta \mathbb{N}$ given by

$$I \longmapsto PIP \qquad \text{and} \qquad J \longmapsto \overline{(A_\infty \times_{\beta_\infty} \mathbb{Z})J(A_\infty \times_{\beta_\infty} \mathbb{Z})}\,.$$

Moreover, if U_∞ is the unitary that implements the automorphism β_∞, then $V_\infty = PU_\infty P$ is the isometry implementing β. Since $\gamma_z(P) = P$ for every $z \in \mathbb{T}$, the canonical gauge action $\gamma : \mathbb{T} \longrightarrow \text{Aut}\,(A_\infty \times_{\beta_\infty} \mathbb{Z})$ restricts to the gauge action of $A \times_\beta \mathbb{N}$.

Lemma 1.4. *If A is a C^*-algebra and $\beta : A \longrightarrow A$ is an extendible and injective endomorphism, then there exists an order preserving bijection between gauge invariant ideals of $A \times_\beta \mathbb{N}$ and $A_\infty \times_{\beta_\infty} \mathbb{Z}$.*

Now, we will describe the gauge invariant ideals in terms of the C^*-dynamical system (A, β). Given an endomorphism $\beta : A \longrightarrow A$, it is easy to check that $\beta(A)$ is a hereditary sub-C^*-algebra of A if and only if $\overline{\beta(A)A\beta(A)} = \beta(A)$.

Definition 1.5. Let A be a C^*-algebra and let $\beta : A \to A$ an endomorphism such that $\beta(A)$ is a hereditary sub-C^*-algebra of A. We say that an ideal I of A is *β-invariant* if $\overline{\beta(A)I\beta(A)} = \beta(I)$. We say that A is *β-simple* if there are no non-trivial β-invariant ideals.

Lemma 1.6. *Let A be a C^*-algebra, and let $\beta : A \to A$ an endomorphism such that $\beta(A)$ is a hereditary sub-C^*-algebra of A. If I is a β-invariant ideal of A, then it is also β^n-invariant for every $n > 0$.*

Proof. Let I be an ideal such that $\overline{\beta(A)I\beta(A)} = \beta(I)$. We will prove the result by induction on n. The case $n = 1$ being clear, suppose that $\overline{\beta^{n-1}(A)I\beta^{n-1}(A)} = \beta^{n-1}(I)$. Observe that, since $\beta(A)$ is a hereditary sub-C^*-algebra of A, we have that $\beta(A) = \overline{\beta(A)A\beta(A)}$. Thus,

$$\begin{aligned}
\overline{\beta^n(A)I\beta^n(A)} &= \overline{\beta^{n-1}(\beta(A)A\beta(A))I\beta^{n-1}(\beta(A)A\beta(A))} \\
&\subseteq \overline{\beta^n(A)\beta^{n-1}(A)I\beta^{n-1}(A)\beta^n(A)} \\
&= \overline{\beta^n(A)\beta^{n-1}(I)\beta^n(A)} = \beta^{n-1}(\overline{\beta(A)I\beta(A)}) = \beta^n(I)\,.
\end{aligned}$$

Therefore, $\overline{\beta^n(A)I\beta^n(A)} = \beta^n(I)$ as desired. □

Remark 1.7. Notice that the converse of the above Lemma is not true in general. Let $A = C_0(\mathbb{Z})$ and let $\beta : C_0(\mathbb{Z}) \to C_0(\mathbb{Z})$ be the automorphism that sends $\chi_{\{i\}}$ (the characteristic function at i) to $\chi_{\{i+1\}}$ for every $i \in \mathbb{Z}$. It is clear that $C_0(\mathbb{Z})$ is β-simple, but $I = C_0(2 \cdot \mathbb{Z})$ is a β^2-invariant ideal.

Observe also that, if I is a β-invariant ideal, then $\beta(I)$ is a hereditary sub-C^*-algebra of A, but the above example also shows that the converse it is not true.

Remark 1.8. Let β be an injective and extendible endomorphism such that $\beta(A)$ is hereditary. If we set the projection $P = \widehat{\varphi_0}(1_{M(A)}) = (1, P_1, P_2, \dots) \in M(A_\infty)$, where $P_n = \widehat{\beta}^n(1_{M(A)})$, then we have that $A \cong \varphi_0(A) = PA_\infty P$. Hence, we can see A as a hereditary sub-C^*-algebra of A_∞ such that $\beta_{\infty|A} = \beta$. Indeed, it is enough to check that, given any $n \in \mathbb{N}$ and $a \in A$, then $P\varphi_n(a)P = \widehat{\varphi_n}(P_naP_n) \in \varphi_0(A)$. But since $P_naP_n \in \overline{\beta^n(A)A\beta^n(A)} = \beta^n(A)$ (by Lemma 1.6), we have that $P\varphi_n(a)P \in \varphi_n(\beta^n(A)) = \varphi_0(A)$.

In [27] Pimsner introduced a class of C^*-algebras (later improved by Katsura [17]) generated by C^*-correspondences (X, φ_X) over A, called Cuntz-Pimsner algebras and denoted by $\mathcal{O}_X$. In particular this class includes crossed products and graph C^*-algebras. Katsura [18] studies gauge-invariant ideals of Cuntz-Pimsner algebras; in particular, when X is a Hilbert A-bimodule (see e.g. [1]), he obtain a bijection between gauge invariant ideals of the Cuntz-Pimsner algebra $\mathcal{O}_X$ and invariant ideals I of A with respect to the correspondence X (i.e., $\varphi_X(I)X = XI$) [18, Theorem 10.6].

Let $\beta : A \longrightarrow A$ is an injective endomorphism such that $\beta(A)$ is a hereditary sub-C^*-algebra. If we set $X := {}_{\beta}A = \beta(A)A$ with left-action φ_X given by the endomorphism β, and right inner product given by $< x, y >_A = x^*y$ for every $x, y \in A$, then we have a C^*-correspondence. We have that $\varphi_X(A) \subseteq \mathcal{K}(X)$ (the compact operators of X), and since $\beta(A)$ is a hereditary sub-C^*-algebra, it follows that $\overline{\beta(A)A\beta(A)} = \beta(A)$, whence $\varphi_X(A) = \mathcal{K}(X)$. Therefore, since β is injective and $\varphi_X(A) = \mathcal{K}(X)$, we can define a left inner product as ${}_A < x, y > := \varphi_X^{-1}(\theta_{x,y})$ for every $x, y \in A$. Hence, X has a natural structure of Hilbert A-bimodule.

Lemma 1.9 (cf. [21])**.** *If A is a C^*-algebra, $\beta : A \longrightarrow A$ is an injective endomorphism such that $\beta(A)$ is a hereditary sub-C^*-algebra of A and $X = {}_{\beta}A$ is the Hilbert A-bimodule defined above, then $\mathcal{O}_X \cong A \times_{\beta} \mathbb{N}$.*

Thus, we can apply Katsura's description of the gauge invariant ideals, and we see that an ideal I of A is invariant with respect to the correspondence X if and only if $\beta(A)I = \beta(I)A$.

Lemma 1.10. *If A is a C^*-algebra and $\beta : A \longrightarrow A$ is an injective endomorphism such that $\beta(A)$ is a hereditary sub-C^*-algebra of A, then I is a β-invariant ideal of A if and only if $\beta(A)I = \beta(I)A$.*

Proof. First, suppose that $\beta(A)I = \beta(I)A$, and observe that $\beta(I) \subseteq I$. Thus we have that

$$\beta(I)A\beta(I) = \beta(A)I\beta(A) \subseteq \overline{\beta(A)A\beta(A)} = \beta(A)\,.$$

Then multiplying at both sides by $\beta(I)$ we have $\beta(A)I\beta(A) = \beta(I)A\beta(I) \subseteq \beta(I)$, and therefore $\overline{\beta(A)I\beta(A)} = \beta(I)$.

In the other side, suppose that $\overline{\beta(A)I\beta(A)} = \beta(I)$. From $\beta(I) \subseteq I$ it follows $\beta(I)A \subseteq \beta(A)I$. Now, let $\{e_n\} \subset I_+$ be an approximate unit of I, and let $a \in A$ and $y \in I$. We claim that $\beta(e_n)\beta(a)y = \beta(e_n a)y$ converges to $\beta(a)y$, whence $\beta(a)y \in \beta(I)A$. Indeed, let $z \in I$ such that $\beta(a)yy^*\beta(a^*) = \beta(z)$. Given $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\|e_n z - z\| < \varepsilon/2$. Then we have

$$\begin{aligned} \|\beta(e_n a)y - \beta(a)y\|^2 &= \|(\beta(e_n a)y - \beta(a)y)(\beta(e_n a)y - \beta(a)y)^*\| \\ &\leq \|\beta(e_n a)yy^*\beta(a^*e_n) - \beta(a)yy^*\beta(a^*e_n)\| + \|\beta(e_n a)yy^*\beta(a^*) - \beta(a)yy^*\beta(a^*)\| \\ &\leq \|\beta(e_n z e_n) - \beta(z e_n)\| + \|\beta(e_n z) - \beta(z)\| \\ &= \|e_n z e_n - z e_n\| + \|e_n z - z\| < \varepsilon/2 + \varepsilon/2 = \varepsilon\,. \end{aligned}$$

Thus $\beta(e_n a)y$ converges to $\beta(a)y$, as desired. $\square$

Proposition 1.11. *If A is a C^*-algebra and $\beta : A \longrightarrow A$ is an injective endomorphism such that $\beta(A)$ is a hereditary sub-C^*-algebra of A, then there is a bijection between gauge*

invariant ideals of $A \times_\beta \mathbb{N}$ and β-invariant ideals of A. Thus, A is β-simple if and only if A_∞ is β_∞-simple.

Proof. First statement holds from [18, Theorem 10.6] and Lemmas 1.9 & 1.10. Last statement follows from [23, Lemma 6.1]. $\square$

Remark 1.12. The bijection stated in Proposition 1.11 sends $I \mapsto \overline{(A \times_\beta \mathbb{N}) \cdot I \cdot (A \times_\beta \mathbb{N})}$ and $K \mapsto K \cap A$.

Finally we give necessary and sufficient conditions for the simplicity of a Stacey crossed product. The main technical device we use is the Connes' spectrum of an endomorphism. This is just a reformulation of the Connes' spectrum for automorphisms (see [22, 16]). We will see that for nice endomorphisms (extendible and hereditary image) the Connes' spectrum of β and that of the associated automorphism β_∞ coincide. Therefore, we will be able to use results by Olesen and Pedersen to determine the conditions for the simplicity of the Stacey crossed products.

Definition 1.13. Let A be a C^*-algebra and let $\beta : A \to A$ be an endomorphism. Then we say that:

(1) β is *inner* if there exists an isometry $W \in M(A)$ such that $\beta = \mathrm{Ad}\, W$.
(2) β is *outer* if it is not inner.
(3) β is *weakly properly outer* if for every β-invariant ideal I and every $n > 0$ the restriction endomorphism $\beta^n_{|I}$ is outer.

Recall [15, Definition 2.1] that an automorphism α of a C^*-algebra I is said to be properly outer if for every nonzero α-invariant two-sided ideal I of A and for every unitary multiplier u of I, $\|\alpha|I - \mathrm{Ad}_u|I\|$. When β is an automorphism, the notion of weakly properly outerness is weaker than the properly outer notion by Elliott [15], later studied by Olesen and Pedersen [25, Theorem 10.4].

Definition 1.14. Let A be a C^*-algebra, let $\beta : A \to A$ be an extendible injective endomorphism and let $\gamma : \mathbb{T} \longrightarrow \mathrm{Aut}\,(A \times_\beta \mathbb{N})$ be the gauge action. We define the *Connes' spectrum* of β as

$$\mathbb{T}(\beta) := \{t \in \mathbb{T} : \gamma_t(I) \cap I \neq 0 \text{ for every } 0 \neq I \lhd A \times_\beta \mathbb{N}\}\,.$$

Remark 1.15. Observe that $\mathbb{T}(\beta)$ is a closed subgroup of $\mathbb{T}$. Hence can only be $\{1\}$, $\mathbb{T}$ or a finite subgroup.

This definition of the Connes' spectrum coincide with the one given by Olesen and Olesen & Pedersen [22, 23] when β is an automorphism. Moreover, using that the bijection between ideals of $A \times_\beta \mathbb{N}$ and these of $A_\infty \times_{\beta_\infty} \mathbb{Z}$ given by

$$I \longmapsto PIP \qquad \text{and} \qquad J \longmapsto \overline{(A_\infty \times_{\beta_\infty} \mathbb{Z})J(A_\infty \times_{\beta_\infty} \mathbb{Z})}\,,$$

and the fact that the canonical gauge action $\gamma : \mathbb{T} \longrightarrow \mathrm{Aut}\,(A_\infty \times_{\beta_\infty} \mathbb{Z})$ restricts to the gauge action of $A \times_\beta \mathbb{N}$ (since $\gamma_z(P) = P$ for every $z \in \mathbb{T}$), the following lemma easily follows.

Lemma 1.16. *If A is a C^*-algebra and $\beta : A \to A$ is an extendible injective endomorphism with $\beta(A)$ being a hereditary sub-C^*-algebra of A, then $\mathbb{T}(\beta) = \mathbb{T}(\beta_\infty)$.*

Let β be an extendible endomorphism such that $\beta = \mathrm{Ad}\, V$, where V is an isometry of $M(A)$. Then we can construct a unitary of $M(A_\infty)$ $U := \sum_{i\geq 0} \widehat{\varphi}_i(V)$ such that $\beta_\infty = \mathrm{Ad}\, U$. Now, let us see a result following from [22].

Theorem 1.17. *Let A be a C^*-algebra and let $\beta : A \to A$ be an extendible injective endomorphism with $\beta(A)$ being a hereditary sub-C^*-algebra of A. Let us consider the following statements:*

(1) $\mathbb{T}(\beta^n) = \mathbb{T}$ *for every* $n > 0$.

(2) *Given* $a \in A^{\sim}$ *(the unitization of A) and any B hereditary sub-C^*-algebra of A, for every $n > 0$ we have that*

$$\mathit{inf}\,\{\|xa\beta^n(x)\| : 0 \leq x \in B \text{ with } \|x\| = 1\} = 0\,.$$

(3) β^n *is outer for every* $n > 0$.

Then, $(1) \Rightarrow (2) \Rightarrow (3)$. *Moreover, if A is β-simple, then* $(3) \Rightarrow (1)$ *(and thus all they are equivalent).*

Proof. $(1) \Rightarrow (2)$ This is [25, Theorem 10.4 and Lemma 7.1]. If $\mathbb{T}(\beta^n) = \mathbb{T}$ then $\mathbb{T}(\beta^n_\infty) = \mathbb{T}$ for every $n > 0$, so β^n_∞ is properly outer for every $n > 0$. Since any hereditary sub-C^*-algebra B of A is also a hereditary sub-C^*-algebra of A_∞, (see Remark 1.8), we can apply [25, Proof of Lemma 7.1] to B. Thus, since $\beta^n_{\infty|A} = \beta^n$, we have the result.

$(2) \Rightarrow (3)$ Suppose that $\beta^n = \mathrm{Ad}\, W$ for an isometry $W \in M(A)$. Fix $\varepsilon > 0$, and take $b \in A_+$ with $\|b\| = 1$. Set $c := f_\varepsilon(b)$, where $f_\varepsilon(t) : [0,1] \longrightarrow \mathbb{R}_+$ is the continuous function that is $f_\varepsilon(0) = 0$, constant 1 for $t \geq \varepsilon$ and linear otherwise. Then, we have that $xc = cx = x$ for every $x \in \overline{(b-\varepsilon)_+ A (b-\varepsilon)_+}$. Hence, given any $0 \leq x \in \overline{(b-\varepsilon)_+ A (b-\varepsilon)_+}$ with $\|x\| = 1$, we have that

$$\begin{aligned}\|x(cW^*)\beta^n(x)\|^2 &= \|x(cW^*)WxW^*\|^2 = \|xcxW^*\|^2\\ &= \|x^2W^*\|^2 = \|x^2W^*Wx^2\| = \|x^4\| = \|x\|^4 = 1\,,\end{aligned}$$

which contradicts the hypothesis, since $cW^* \in A$.

Now, suppose that A is β-simple. We are going to prove that $(3) \Rightarrow (1)$. By [25, Theorem 10.4] we have that $\mathbb{T}(\beta_\infty) = \mathbb{T}$ if and only if $\mathbb{T}(\beta^n_\infty) = \mathbb{T}$ for every $n \in \mathbb{N}$. Let us suppose that $\mathbb{T}(\beta) = \mathbb{T}(\beta_\infty) \neq \mathbb{T}$. Hence, $\mathbb{T}(\beta_\infty)$ is a finite subgroup, and thus the complement $\mathbb{T}(\beta_\infty)^\perp \neq \{0\}$. Therefore, by [25, Theorem 4.5], for every $k \in \mathbb{T}(\beta_\infty)^\perp$ we have that $\beta^k_\infty = \mathrm{Ad}\, U$, where $U \in M(A_\infty)$. But then $V = PUP \in M(A)$ is an isometry such that $\beta^k = \mathrm{Ad}\, V$, a contradiction. □

Then it follows the characterization of simplicity.

Corollary 1.18. *Let A be a C^*-algebra and let $\beta : A \to A$ be an extendible injective endomorphism with $\beta(A)$ being a hereditary sub-C^*-algebra of A. Then $A \times_\beta \mathbb{N}$ is simple if and only if A is β-simple and β^n is outer for every $n > 0$.*

Proof. $A \times_\beta \mathbb{N}$ is simple if and only if $A_\infty \times_{\beta_\infty} \mathbb{Z}$ is simple if and only if A_∞ is β_∞-simple and $\mathbb{T}(\beta_\infty) = \mathbb{T}$ [23, Theorem 6.5] if and only if A is β-simple and $\mathbb{T}(\beta) = \mathbb{T}$. Therefore, by Theorem 1.17 we have that A is β-simple and $\mathbb{T}(\beta) = \mathbb{T}$ if and only if A is β-simple and β^n is outer for every $n > 0$. □

Example 1.19. The following example is [25, Theorem 9.1]. Let A be a C^*-algebra with a faithful bounded trace, let $A \longrightarrow \mathcal{B}(\mathcal{H})$ be a faithful non-degenerate representation of A, and let V be a non-unitary isometry of $\mathcal{B}(\mathcal{H})$ with $VV^* \in M(A)$ such that $VAV^* + V^*AV \subseteq A$. Then suppose that there are no non-trivial ideals I of A such that $VIV^* + V^*IV \subseteq I$. Then we claim that the C^*-algebra $B := C^*(\{AV^n(V^*)^m\}_{n,m\geq 0}) \subseteq \mathcal{B}(\mathcal{H})$ is simple. Indeed, let us define the endomorphism $\beta : A \longrightarrow A$ by $\beta(a) = VaV^*$ for every $a \in A$ that is extendible (since $VV^* \in M(A)$). Clearly satisfies that $\beta(A)$ is a hereditary sub-C^*-algebra of A, and it does not have any non-trivial β-invariant ideal. Now, since τ is a faithful bounded trace of A, we can extended it to a faithful bounded trace $\bar{\tau}$ of $M(A)$. Hence, $M(A)$ has no non-unitaries isometries. Therefore, by Theorem 1.17 $A \times_\beta \mathbb{N}$ is simple, whence the natural map $A \times_\beta \mathbb{N} \longrightarrow B$ is an isomorphism.

2. Graph C^*-algebras

In this section, we apply the above results to determine the simplicity of certain graph C^*-algebras. Though their simplicity is well understood in terms of properties of the graph, we are going to deduce it from the properties of their associated C^*-dynamical systems.

We use the conventions of [28]. Let $E = (E^0, E^1, r, s)$ be a countable directed graph; $r, s : E^1 \to E^0$ denote the *range* and the *source* maps of an edge. We say that E is *column-finite* if $|s^{-1}(v)| < \infty$ for every $v \in E^0$. A vertex $v \in E^0$ is a *sink* (*source*) if $|s^{-1}(v)| = 0$ ($|r^{-1}(v)| = 0$). A vertex $v \in E^0$ is called *singular* if is either a source or an infinite receiver. We denote by E^0_{sing} the set of all singular vertices. A *path* α of length n is a concatenation of n edges $e_n \cdots e_1$ with $r(e_i) = s(e_{i+1})$ for $i = 1, ..., n-1$. Given a path α we denote by $|\alpha|$ its length. Let E^n be the set of all paths of length n, and $E^* = \cup_{n\geq 0} E^n$ the set of all the paths of finite length in E. Finally, given $\alpha, \eta \in E^*$, we say that $\alpha \in \eta$ if there exist $\rho, \gamma \in E^*$ such that $\eta = \rho\alpha\gamma$.

Recall that the *graph C^*-algebra* $C^*(E)$ is the universal C^*-algebra generated by orthogonal projections $\{P_v\}_{v\in E^0}$ and partial isometries $\{S_e\}_{e\in E^1}$, satisfying the following conditions:

$$(CK1) \qquad S_e^* S_f = \delta_{e,f} \cdot P_{s(e)} \qquad \text{for every } e, f \in E^1$$
$$(CK2) \qquad P_v = \sum_{r(e)=v} S_e S_e^* \qquad \text{for every } v \in E^0_{sing}.$$

See [28] for a survey on graph C^*-algebras. One can naturally define a group homomorphism $\gamma : \mathbb{T} \to \mathrm{Aut}\ C^*(E)$, given by $\gamma_z(P_v) = P_v$ and $\gamma_z(S_e) = zS_e$ for every $z \in \mathbb{T}$, $v \in E^0$ and $e \in E^1$; it is the so-called *gauge action* on $C^*(E)$. An ideal I of $C^*(E)$ is said to be a *gauge invariant ideal* if $\gamma_z(I) = I$ for every $z \in \mathbb{T}$ (see [8] and [9]). The *core* sub-C^*-algebra of $C^*(E)$ is defined as

$$C^*(E)^\gamma := \{x \in C^*(E) : \gamma_z(x) = x \text{ for every } z \in \mathbb{T}\}.$$

We can give another description of the core. For every $n \in \mathbb{N}$ and $v \in E^0$, define

$$\mathcal{F}_n(v) := \{S_\eta S_\rho^* : \eta, \rho \in E^n \text{ with } s(\eta) = s(\rho) = v\} \cong M_{k_{n,v}}(\mathbb{C})$$

for some $k_{n,v} \in \mathbb{N}$, and let $\mathcal{F}_n = \oplus_{v\in E^0} \mathcal{F}_n(v)$. Now, if we index the vertices $\{v_i\}_{i\geq 0}$, then we define $C_{n.m} := \sum_{i,j\geq 0}^{n,m} \mathcal{F}_i(v_j)$ for every $n, n \geq 0$. These are finite dimensional sub-C^*-algebras

of $C^*(E)^\gamma$ with $C_{n,m} \subseteq C_{n,m+1}$ and $C_{n,m} \subseteq C_{n+1,m}$ for every $n, m \geq 0$. Hence,

$$C^*(E)^\gamma = \overline{\bigcup_{n,m\geq 0} C_{n,m}}$$

is an AF-algebra.

We recall the following result from [4], that allows to present certain graph C^*-algebras as C^*-dynamical systems (we would like to thank the authors for showing us the result even before the releasing of the manuscript).

Theorem 2.1 ([4, Theorem 9.3])**.** *Let E be a column finite graph without sinks. If we define the endomorphism $\beta_E : C^*(E)^\gamma \to C^*(E)^\gamma$ as $\beta_E(z) = TzT^*$ for every $z \in C^*(E)^\gamma$, where*

$$T = \sum_{e\in E^1} |s^{-1}(s(e))|^{1/2} S_e$$

is an isometry of $M(C^*(E))$, then we have that $C^*(E) \cong C^*(E)^\gamma \times_{\beta_E} \mathbb{N}$.

Notice that the endomorphism $\beta_E : C^*(E)^\gamma \to C^*(E)^\gamma$ is injective, extendible and $\beta_E(C^*(E)^\gamma)$ is a hereditary sub-C^*-algebra of $C^*(E)^\gamma$.

Definition 2.2. A subset $H \subseteq E^0$ is said to be *hereditary* if, whenever $\eta \in E^*$ with $r(\eta) \in H$, then $s(\eta) \in H$. We say that H is *saturated* if, whenever $|r^{-1}(v)| < \infty$ and $\{s(r^{-1}(v))\} := \{z \in E^0 : z = s(e) \text{ for some } e \in r^{-1}(v)\} \subseteq H$, then $v \in H$.

By [8, Theorem 4.1], there exists a bijection between hereditary and saturated subsets of E^0 and gauge invariant ideals of $C^*(E)$, $H \longmapsto K_H$, where $K_H := \overline{\text{span}}\{S_\eta S_\nu^* : \eta, \nu \in E^* \text{ with } s(\eta) = s(\nu) \in H\}$. The inverse map is $K \longmapsto H_K$, where $H_K := \{v \in E^0 : P_v \in K\}$.

Now, given a hereditary and saturated subset of E^0, we define

$$I_H := K_H \cap C^*(E)^\gamma.$$

By Remark 1.12, I_H is a β_E-invariant ideal of $C^*(E)^\gamma$, and it is easy to see that

$$I_H := \overline{\sum_{v\in H, n\geq 0} \mathcal{F}_n(v)}.$$

On the other side, if K is a gauge invariant ideal of $C^*(E)$, since $I := K \cap C^*(E)^\gamma$ is a β_E-invariant ideal, then we have that the set

$$H_I := \{v \in E^0 : P_v \in I\}$$

is a subset of H_K. Moreover, since $P_v \in C^*(E)^\gamma$ for every $v \in E^0$, it is clear that $H_K \subseteq H_I$, whence $H_K = H_I$. Thus, H_I is an hereditary and saturated subset of E^0.

In particular, if I is a β_E-invariant ideal of $C^*(E)^\gamma$, then $K := \overline{(C^*(E)) \cdot I \cdot (C^*(E))}$ is a gauge invariant ideal of $C^*(E)$ and

$$I_{H_I} = I_{H_K} = K_{H_K} \cap C^*(E)^\gamma = K \cap C^*(E)^\gamma = I.$$

Conversely, if H is a hereditary and saturated subset of E^0, since $I_H := K_H \cap C^*(E)^\gamma$, we conclude that

$$H_{I_H} = H_{K_H} = H.$$

Summarizing, there exists a bijection between the hereditary and saturated subsets of E^0 and the β_E-invariant ideals of $C^*(E)^\gamma$ defined by the maps

$$H \longmapsto I_H = \overline{\sum_{v\in H, n\geq 0} \mathcal{F}_n(v)} \qquad \text{and} \qquad I \longmapsto H_I = \{v \in E^0 : P_v \in I\}.$$

One could be tempted to think that there is a bijection between hereditary sets of E^0 and the ideal of $C^*(E)^\gamma$ such that $\beta_E(I) \subseteq I$, but this is not the case (see Examples 2.5).

Theorem 2.3 (cf. [8, Theorem 4.1])**.** *If E is a column finite graph without sinks, then there is a bijection between the closed gauge invariant ideals of $C^*(E)$, the hereditary and saturated subsets of E^0 and the β_E-invariant ideals of $C^*(E)^\gamma$.*

Corollary 2.4. *Let E be a column finite graph without sinks, then E^0 has no non-trivial hereditary and saturated subsets if and only if $C^*(E)^\gamma$ does not have a proper β_E-invariant ideals.*

Example 2.5. In the following examples we would like to illustrate some consequences of Corollaries 1.18 and 2.4 and determine the simplicity of some graph C^*-algebras. We would like to remark again that this is well-known by [8, Proposition 5.1]. However, one can slightly modify some of the examples to get new simple C^*-algebra that probably do not arise as graph C^*-algebras.

(1) Consider the graph E

$$\bullet_{v_0} \longrightarrow \bullet_{v_1} \longrightarrow \bullet_{v_2} \longrightarrow \cdots$$

Then E^0 has no non-trivial hereditary and saturated subsets. We have that $C^*(E)^\gamma \cong C_0(\mathbb{N}\cup\{0\})$ and the endomorphism $\beta_E : C^*(E)^\gamma \to C^*(E)^\gamma$ sends $\chi_{\{i\}}$ (the characteristic function at i) to $\chi_{\{i+1\}}$ for every $i \geq 0$. Then, since E^0 does not have non-trivial saturated and hereditary subsets, $C_0(\mathbb{N}\cup\{0\})$ is β_E-simple. Moreover, since $M(C_0(\mathbb{N}\cup\{0\}))$ is a commutative C^*-algebra, it does not have non-unitary isometries. Hence, $C^*(E)$ is simple.

(2) Consider the graph E

$$\overset{\overset{e}{\circlearrowright}}{\bullet}_{v_0} \longrightarrow \bullet_{v_1} \longrightarrow \bullet_{v_2} \longrightarrow \cdots$$

Then $C^*(E)^\gamma = \mathcal{K}$ (the compact operators of a countable infinite dimensional Hilbert space $\mathcal{H}$), that is simple. Therefore $C^*(E)^\gamma$ is (β_E-) simple, and thus E^0 has no non-trivial hereditary and saturated subsets. Moreover, it is not difficult to see that $\beta_E = \mathrm{Ad}\, W$ where W is the shift operator of $\mathcal{H}$, whence β_E is inner and $C^*(E)$ is not a simple C^*-algebra.

(3) This is the graph C^*-algebra picture of the algebra $\mathcal{O}_n$. Let E be the graph

$$\overset{\overset{(n)}{\circlearrowright}}{\bullet}_{v}$$

with n loops. We have that $C^*(E)^\gamma$ is isomorphic to the n-infinity UHF-algebra $\mathcal{U}_n := \bigotimes_{i=1}^{\infty} M_n$ and $\beta_E(x) = P \otimes x$ for every $x \in \mathcal{U}_n$, where $P = \begin{pmatrix} 1/n & \cdots & 1/n \\ \vdots & & \vdots \\ 1/n & \cdots & 1/n \end{pmatrix}$. Therefore $C^*(E)^\gamma$ is β_E-simple, since $\mathcal{U}_n$ is simple. Moreover, since $C^*(E)^\gamma$ is a unital and finite C^*-algebra it does not have non-unitary isometries, and therefore $C^*(E)$ is a simple C^*-algebra.

(4) An example of the different behaviour of β_E and β_E^2 can be found when the graph E is

$$\begin{tikzcd} \bullet_v \arrow[r, shift left=2] \arrow[r, shift left] & \bullet_w \arrow[l, shift left] \end{tikzcd}$$

In this case P_v and P_w generate two orthogonal ideals of $C^*(E)^\gamma$, I_v and I_w respectively, both isomorphic to the CAR-algebra $\bigotimes_{n=1}^{\infty} M_2$, and such that $C^*(E)^\gamma = I_v \oplus I_w$. We have that $\beta_E(x, y) = (y, P \otimes x)$ for every $(x, y) \in I_v \oplus I_w$, where $P = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}$. Therefore $C^*(E)^\gamma$ is β_E-simple, but $\beta_E^2(I_v) \subseteq I_v$ and $\beta_E^2(I_w) \subseteq I_w$. Moreover, since $C^*(E)^\gamma$ is a unital and finite C^*-algebra, it does not have non-unitary isometries, and therefore $C^*(E)$ is a simple C^*-algebra.

(5) Consider the graph E

$$\begin{tikzcd} \bullet_v \arrow[r, "f"'] & \bullet_w \arrow[loop above, "e"] \end{tikzcd}$$

Observe that $C^*(E)^\gamma = \overline{\operatorname{span}}\,\{P_v,\, S_{e^m} S_{e^m}^*,\, S_{e^m f} S_{e^m f}^* : m \geq 0\}$. Then, $C^*(E)^\gamma$ is a commutative C^*-algebra isomorphic to $C(X)$ where $X = \{1/n : 1 \leq n\} \cup \{0\}$, and the endomorphism acts $\beta_E(\chi_{[0,1/n]}) = \chi_{[0,1/(n+1)]}$ for every $n \geq 1$. Since $\{v\}$ is a saturated and hereditary subset, there exists a proper β_E-invariant ideal, that corresponds to the ideal $C_0(X \setminus \{0\})$. Given $n \in \mathbb{N}$ let I_n be the ideal of $C^*(E)^\gamma$ generated by $\chi_{[0,1/n]}$. Observe that $\beta_E(I_n) \subseteq I_n$ (in particular $\beta_E(I_n) = I_{n+1}$). Therefore $C^*(E)^\gamma$ posses a infinite countably family of different ideals I such that $\beta_E(I) \subseteq I$.

Definition 2.6. A C^*-dynamical system (A, β) is said to satisfy the *Cuntz-Krieger uniqueness theorem* if for every faithful Stacey covariant representation (π, V) of (A, β) we have that $C^*(\pi, V) \cong A \times_\beta \mathbb{N}$.

Recall that the graph E satisfies *condition (L)* if every cycle has an entry. A graph E satisfies condition (L) if and only if, given any $*$-homomorphism $\eta : C^*(E) \to B$ such that $\eta(P_v) \neq 0$ for every $v \in E^0$, we have that η is injective (see e.g. [28, Section 2]). Thus,

Theorem 2.7 (cf. [28, Theorem 2.4] & [24, Theorem 2.5])**.** *Let E be a column finite graph without sinks. Then the following statements are equivalent:*

(1) The graph E satisfies condition (L).
(2) $(C^*(E)^\gamma, \beta_E)$ satisfies the Cuntz-Krieger uniqueness theorem.
(3) $\mathbb{T}(\beta_E) = \mathbb{T}$.

Now we will see that for the dynamical system $(C^*(E)^\gamma, \beta_E)$ associated to a graph C^*-algebras, the results of Olesen and Pedersen [24, Theorem 2.5 & Theorem 4.6] reduces to a simpler way.

Proposition 2.8. *Let E be a column finite graph without sinks and let $(C^*(E)^\gamma, \beta_E)$ be its associated C^*-dynamical system. If β_E is weakly properly outer then E satisfies condition (L).*

Proof. Suppose that β_E is weakly properly outer and that E does not satisfy condition (L), i.e., there exists a cycle α without an entry. We can suppose that $\alpha = e_n \cdots e_1$ with $e_i \in E^1$ and $v_i = r(e_i)$ for $i = 1, \dots, n$, such that $s(e_1) = r(e_n) = v_n$ and $r(e_i) \neq v_n$ for every $i \neq n$. Let $H_\alpha = \{v_i\}_{i=1,\dots,n}$, and let I be the ideal of $C^*(E)^\gamma$ generated by $\{P_v\}_{v \in H_\alpha}$. Observe that, since α does not have any exit, by (CK2) we have that

$$I = \overline{\sum_{k \geq 0} \mathcal{F}_k(v_n)}\,.$$

Given $w \in E^0$, let $\{\eta_{i,w}\}_{i=1}^{\nu_w} \subseteq E^n$ be the set of paths such that $s(\eta_{i,w}) = w$ (a finite number since E is column finite),. Given any $z \in E^0_{v_n}$, where

$$E^0_{v_n} := \{z \in E^0 : \text{ exists } \eta \in E^* \text{ with } s(\eta) = v_n \text{ and } r(\eta) = z\}\,,$$

consider all the paths $\{\gamma_{j,z}\}_{j \in \Delta_z} \subseteq E^*$ such that $s(\gamma_{j,z}) = v_n$ and $r(\gamma_{j,z}) = z$. Observe that $1 \leq |\{\gamma_{j,z}\}_{j \in \Delta_z}| \leq \infty$. Given any path $\gamma_{j,z}$, we define $\kappa_{i,z} := |s^{-1}(s(f_n))| \cdots |s^{-1}(s(f_1))| < \infty$ for $f_n \cdots f_1 = \eta_{i,z}$ with $f_i \in E^1$. Then, define the formal sums (we still not determine where their converge to)

$$V_w := \sum_{j \in \Delta_w, i=1}^{\nu_w} \kappa_{i,w}^{-1/2} S_{\eta_{i,w}\gamma_{j,w}} S^*_{\gamma_{j,w}\alpha} \qquad \text{if } w \in E^0_{v_n} \setminus H_\alpha\,,$$

$$V_{v_k} = \sum_{i=1}^{\nu_{v_k}} \kappa_{i,v_k}^{-1/2} S_{\eta_{i,v_k} e_k \cdots e_1} S^*_{e_k \cdots e_1 \alpha} \qquad \text{for } 1 \leq k \leq n-1$$

and

$$V_{v_n} = \sum_{i=1}^{\nu_{v_n}} \kappa_{i,v_n}^{-1/2} S_{\eta_{i,v_n}} S^*_\alpha\,.$$

We claim that $\sum_{w \in E^0_{v_n}} V_w$ converges with the strong topology in $M(I)$. Indeed, recall that $I = \overline{\text{span}}\{S_{\gamma_{i,w}} S^*_{\gamma_{j,z}} : |\gamma_{i,w}| = |\gamma_{j,z}| \text{ for } z, w \in E^0_{v_n}\}$, so it is enough to see that for every $v, w \in E^0_{v_n}$ and $k, l \in \mathbb{N}$ such that $|\gamma_{k,w}| = |\gamma_{l,z}|$ then $(\sum_{u \in E^0_{v_n}} V_u) S_{\gamma_{k,w}} S^*_{\gamma_{l,z}}$ and $S_{\gamma_{k,w}} S^*_{\gamma_{l,z}} (\sum_{u \in E^0_{v_n}} V_u)$ are elements of I of norm less or equal to 1. Observe that

$$(\sum_{u \in E^0_{v_n}} V_u) S_{\gamma_{k,w}} S^*_{\gamma_{l,z}} = V_w S_{\gamma_{k,w}} S^*_{\gamma_{l,z}} = \sum_{i=1}^{\nu_w} \kappa_{i,w}^{-1/2} S_{\eta_{i,w}\gamma_{k,w}} S^*_{\gamma_{l,z}\alpha} \in I\,.$$

Now we have that

$$\begin{aligned} \|\sum_{i=1}^{\nu_w} \kappa_{i,w}^{-1/2} S_{\eta_{i,w}\gamma_{k,w}} S^*_{\gamma_{l,z}\alpha}\|^2 &= \|(\sum_{i=1}^{\nu_w} \kappa_{i,w}^{-1/2} S_{\eta_{i,w}\gamma_{k,w}} S^*_{\gamma_{l,z}\alpha})^* \sum_{i=1}^{\nu_w} \kappa_{i,w}^{-1/2} S_{\eta_{i,w}\gamma_{k,w}} S^*_{\gamma_{l,z}\alpha}\| \\ &= \|\sum_{i=1}^{\nu_w} \kappa_{i,w}^{-1} S_{\gamma_{l,z}\alpha} S^*_{\gamma_{l,z}\alpha}\| \leq \|S_{\gamma_{l,z}} S^*_{\gamma_{l,z}}\| = 1 \end{aligned}$$

as desired. Analogously, we can see that $S_{\gamma_{k,w}} S^*_{\gamma_{l,z}} (\sum_{u \in E^0_{v_n}} V_u)$ converges to an element of I of norm less or equal than 1. Define $V := \sum_{w \in E^0_{v_n}} V_w$. Then we have that $V^* V = 1_{M(I)}$. One

can easily check that $VzV^* = {\beta_E}^n(z)$ for every $z \in I$, so ${\beta_E}^n_{|I}$ is inner, a contradiction with the hypothesis. $\square$

Remark 2.9. In the proof of Proposition 2.8 we prove that, if E does not satisfy condition (L), i.e., there exists a loop α without entries, then given any vertex $v \in E^0$ of the loop α we have that

$$I := \overline{\sum_{n\geq 0} \mathcal{F}_k(v)}\,,$$

is a β_E-invariant ideal of $C^*(E)^\gamma$ such that ${\beta_E}^n_{|I}$ is inner for some $n > 0$. Then H_I is a hereditary and saturated subset of E^0 containing $H_\alpha := \{v \in E^0 : v \in \alpha\}$ such that $I = I_{H_I}$. But since $I_{H_I} \subseteq I_H$ for every hereditary and saturated subset H of E^0 containing H_α we have that H_I is the minimal hereditary and saturated subset of E^0 containing H_α. Thus I is a minimal β_E-invariant ideal such that ${\beta_E}^n_{|I}$ is inner.

Observe also that in general I is a non-simple sub-C^*-algebra of $C^*(E)^\gamma$. For example, if $n > 1$, then the ideal generated by P_{v_i} for every $1 \leq i \leq n$ is proper (one can check easily that it cannot contain P_{v_j} for $i \neq j$).

Proposition 2.10. *Let E be a column finite graph without sinks and let $(C^*(E)^\gamma, \beta_E)$ be its associated C^*-dynamical system. If E satisfies condition (L) then β_E is weakly properly outer.*

Proof. Suppose that E satisfies condition (L) and there exist a non-zero β_E-invariant ideal I of $C^*(E)^\gamma$ such that $\beta^n_{E|I}$ is inner for some $n > 0$. Hence, $I = \overline{\sum_{v\in H_I, k\geq 0} \mathcal{F}_k(v)}$, where H_I is a hereditary and saturated subset of E^0. So, there is a gauge-invariant ideal of $C^*(E)$, say K_{H_I}, generated by $\{P_v\}_{v\in H_I}$. Recall that the core $K^\gamma_{H_I}$ is precisely I. Now, there exists an isometry $W \in M(I)$ such that ${\beta_E}^n(z) = WzW^*$ for every $z \in I$. Since I contains an approximate unit for K_{H_I} (see for example [5, Lemma 3.4]) we can see $M(I)$ as a sub-C^*-algebra of $M(K_{H_I})$.

Define $U := W^*T^n$ a unitary in $M(K_{H_I})$ (we are also using that $TI \subseteq IT$). Then, for every $z \in I$ we have $z = UzU^*$ Observe that, given any $y \in I$, we have that that $yU = Uy$.

We claim that $H_I \subseteq E^0$ cannot have sources. Indeed, if $v \in H_I$ is a source, then $P_v \in I$, and hence

$$P_v = UP_vU^* = W^*T^nP_vT^{n*}W\,.$$

We have that $W^*T^nP_v = \sum_l \lambda_l S_{\eta_l} S^*_{\rho_l} \in P_vK_{H_I}P_v \subseteq P_vC^*(E)P_v$, with $\lambda_l \in \mathbb{C}$ and $\eta_l, \rho_l \in E^*$ with $r(\eta_l) = r(\rho_l) = v$ and $|\eta_l| = |\rho_l| + n$. But this contradicts the fact that v is a source.

Now, given any $v \in E^0$, let $\{\eta_{i,v}\}_{i=1}^{\nu_v}$ be the set of all the paths in E^n with $r(\eta_{i,v}) = v$. So, given $v \in H_I$ and $i \leq \nu_v$, we define

$$0 \neq X_{i,v} := S^*_{\eta_{i,v}} U \in P_{s(\eta_{i,v})} C^*(E)^\gamma P_v\,.$$

If $\mu, \gamma \in E^m$ with $s(\mu) = s(\gamma) \in H_I$ then we have that $S_\mu S^*_\gamma \in I$. So, for every $w, z \in H_I$, $1 \leq k \leq \nu_w$ and $1 \leq l \leq \nu_z$, we have that

(1) $X_{k,w}(S_\mu S^*_\gamma)X^*_{l,z} = S^*_{\eta_{k,w}} U(S_\mu S^*_\gamma)U^*S^*_{\eta_{l,z}} = S^*_{\eta_{k,w}}(S_\mu S^*_\gamma)S_{\eta_{l,z}}$,

(2) $X_{k,w}X^*_{l,z} = \delta_{w,z} \cdot \delta_{k,l} \cdot P_{s(\eta_{k,w})}$,

(3) $X^*_{k,w}S_\mu S^*_\gamma X_{l,z} = U^*S_{\eta_{k,w}} S_\mu S^*_\gamma S^*_{\eta_{l,z}} U = \delta_{r(\mu),s(\eta_{k,w})} \cdot \delta_{r(\gamma),s(\eta_{j,z})} \cdot S_{\eta_{k,w}} S_\mu S^*_\gamma S^*_{\eta_{l,z}}$,

while given $v \in H_I$ and $1 \leq i \leq \nu_v$, we have that

$$X_{i,v}X^*_{i,v}X_{i,v} = S^*_{\eta_{i,v}} UU^*S_{\eta_{i,v}} S^*_{\eta_{i,v}} U = S^*_{\eta_{i,v}} S_{\eta_{i,v}} S^*_{\eta_{i,v}} U = S^*_{\eta_{i,v}} U = X_{i,v}\,,$$

so $X_{i,v}$ is a partial isometry in $P_{s(\eta_{i,v})}C^*(E)^\gamma S_{\eta_{i,v}}S^*_{\eta_{i,v}}$.

Now, choose any $v \in H_I$ and $1 \leq i \leq \nu_v$, and consider the isometry $X_{i,v}S_{\eta_{i,v}} \in P_{s(\eta_{i,v})}C^*(E)P_{s(\eta_{i,v})}$. Given any $\varepsilon > 0$, there exist $m \in \mathbb{N}$, $\lambda_j \in \mathbb{C} \setminus \{0\}$ and $\alpha_j, \beta_j \in E^*$ with $|\alpha_j| = |\beta_j| + n$ and $r(\alpha_j) = r(\beta_j) = s(\eta_{i,v})$ such that

$$\|X_{i,v}S_{\eta_{i,v}} - \sum_{j=1}^{m} \lambda_j S_{\alpha_j} S^*_{\beta_j}\| < \varepsilon\,.$$

Suppose that $|\beta_1| \geq |\beta_i|$ for every $i \leq m$. Then we have that

$$\begin{aligned}
\|p_{s(\eta_{i,v})} - y^*y\| &= \|(X_{i,v}S_{\eta_{i,v}})^* X_{i,v}S_{\eta_{i,v}} - y^*y\| \\
&\leq \|(X_{i,v}S_{\eta_{i,v}})^* X_{i,v}S_{\eta_{i,v}} - y^*X_{i,v}S_{\eta_{i,v}}\| + \|y^*X_{i,v}S_{\eta_{i,v}} - y^*y\| \\
&\leq \|(X_{i,v}S_{\eta_{i,v}})^* - y^*\|\|X_{i,v}S_{\eta_{i,v}}\| + \|X_{i,v}S_{\eta_{i,v}} - y\|\|y^*\| \\
&\leq \varepsilon \cdot 1 + \varepsilon(1+\varepsilon)
\end{aligned}$$

Thus, if $\varepsilon < 1/4$, we have that y^*y is invertible in $P_{s(\eta_{i,v})}C^*(E)^\gamma P_{s(\eta_{i,v})}$. Hence, $yS_{\beta_j} \neq 0$ for every $1 \leq j \leq m$. Thus,

$$\begin{aligned}
\|S_{\beta_1}S^*_{\beta_1} - yS_{\beta_1}S^*_{\beta_1}y^*\| &= \|(X_{i,v}S_{\eta_{i,v}})S_{\beta_1}S^*_{\beta_1}(X_{i,v}S_{\eta_{i,v}})^* - yS_{\beta_1}S^*_{\beta_1}y^*\| \\
&\leq \|(X_{i,v}S_{\eta_{i,v}})S_{\beta_1}S^*_{\beta_1}(X_{i,v}S_{\eta_{i,v}})^* - (X_{i,v}S_{\eta_{i,v}})S_{\beta_1}S^*_{\beta_1}y^*\| + \\
&\quad + \|(X_{i,v}S_{\eta_{i,v}})S_{\beta_1}S^*_{\beta_1}y^* - yS_{\beta_1}S^*_{\beta_1}y^*\| \\
&\leq \|(X_{i,v}S_{\eta_{i,v}})S_{\beta_1}S^*_{\beta_1}\|\varepsilon + \|S_{\beta_1}S^*_{\beta_1}y^*\|\varepsilon \\
&\leq \varepsilon + \varepsilon(1+\varepsilon)
\end{aligned}$$

and therefore $yS_{\beta_1}S^*_{\beta_1}y^*$ is invertible in $S_{\beta_1}S^*_{\beta_1}C^*(E)^\gamma S_{\beta_1}S^*_{\beta_1}$. So, $0 \neq S^*_{\beta_1}yS_{\beta_1} = \sum_{j=1}^{m'} \lambda'_j S_{\gamma_j}$, where $\gamma_j \in E^n$ with $s(\gamma_j) = r(\gamma_j) = r(\beta_1) = s(\eta_{i,v})$ for every $1 \leq j \leq m'$. Hence, the γ_js are cycles. Let $\gamma = \gamma_1$. Since by assumption E satisfies condition (L), we have that γ has an entry. Therefore, there exists $\eta \in E^*$ such that $\gamma \notin \eta$. So, we have that $S_{\beta_1\eta}S^*_{\beta_1\eta} \in S_{\beta_1}S^*_{\beta_1}C^*(E)^\gamma S_{\beta_1}S^*_{\beta_1}$ and hence $S^*_{\beta_1\eta}S_{\beta_1} = S^*_\eta S_\gamma = 0$, that contradicts the fact that $S_{\beta_1}S^*_{\beta_1}$ is invertible in $S_{\beta_1}S^*_{\beta_1}C^*(E)^\gamma S_{\beta_1}S^*_{\beta_1}$. □

Summarizing, we have the following result.

Theorem 2.11. *Let E be a column finite graph without sinks. Then the following statements are equivalent:*

(1) The graph E satisfies condition (L).
(2) $(C^*(E)^\gamma, \beta_E)$ satisfies the Cuntz-Krieger uniqueness theorem.
(3) $\mathbb{T}(\beta_E) = \mathbb{T}$.
(4) There is no β_E-invariant ideal I of $C^*(E)^\gamma$ and $n \in \mathbb{N}$ such that $\beta^n_{E|I} = Ad\,V$, where $V \in M(I)$ is an isometry.
(5) β_E is weakly properly outer.

Finally, using the characterization of simplicity of $C^*(E)$ in terms of properties of the graph E [8, Proposition 5.1], the representation of an Huef and Raeburn of the graph C^*-algebra $C^*(E)$ as the Stacey crossed product $C^*(E)^\gamma \times_{\beta_E} \mathbb{N}$, joint with Corollary 1.18 and Theorem 2.11, we conclude the desired result.

Theorem 2.12 (cf. [8, Proposition 5.1]). *Let E be a column finite graph without sinks. Then the following statements are equivalent:*

(1) $C^*(E)^\gamma \times_{\beta_E} \mathbb{N}$ is simple.
(2) E does not have non-trivial hereditary and saturated subsets and satisfies condition (L).
(3) $C^*(E)^\gamma$ does not have any proper β_E-invariant ideal and β_E^n is outer for every $n \geq 1$.

3. Pure infiniteness

In Theorem 1.17 we have given necessary and sufficient conditions on the endomorphism β for the simplicity of the C^*-algebra $A \times_\beta \mathbb{N}$. If A is a unital C^*-algebra and $\beta(1) \neq 1$ we have then that $A \times_\beta \mathbb{N}$ contains a proper isometry, and if in addition $A \times_\beta \mathbb{N}$ is simple, we have that it is a properly infinite C^*-algebra. We will see that for a broad class of unital real rank zero C^*-algebras A we have that $A \times_\beta \mathbb{N}$ turns out to be purely infinite. Our result generalize and unify similar results given in [29] and [16].

Lemma 3.1. *Let A be a unital C^*-algebra, let $\beta : A \longrightarrow A$ be an injective endomorphism, and suppose that does not exist any proper ideal I of A such that $\beta(I) \subseteq I$. Then, given any non-zero $a \in A_+$ there exists $n \in \mathbb{N}$ such that $a + \beta(a) + \cdots + \beta^n(a)$ is a full positive element in A.*

Proof. Consider the ideal $I := \overline{\operatorname{span}}\,\{x\beta^n(a)y : n \geq 0,\, x, y \in A\} \neq 0$. It clearly satisfies $\beta(I) \subseteq I$ and then, by hypothesis, we have that $I = A$. Therefore we can write

$$1 = \sum_{i=1}^{k} x_i \beta^{n_k}(a) y_k$$

where $x_i, y_i \in A$ and $n_i \in \mathbb{N}$ for every $i \in \{1, \ldots, k\}$. Then, taking $n = \max_i\{n_i\}$, we have or desired result. □

Let $T(A)$ be the set of tracial states of A, which is a compact space with the $*$-weak topology. We say that A has *strict comparison* if: (i) $T(A) \neq \emptyset$; (ii) Whenever $p \in \overline{AqA}$ such that $\tau(p) < \tau(q)$ for every $\tau \in T(A)$, we have that $p \lesssim q$. For example, every unital exact and stably finite C^*-algebra of real rank zero that is $\mathcal{Z}$-stable has strict comparison [30, Corollary 4.10].

Recall that a (non-necessarily simple) C^*-algebra A is said to be *purely infinite* if and only if all positive elements are properly infinite [19] ; in particular, every projection of A (if it has any) must be properly infinite. Also recall that a unital simple C^*-algebra is purely infinite if and only if has real rank zero and every projection is infinite [35]. The following lemma is a slight modification of [29, Lemma 3.2].

Lemma 3.2 (cf. [29, Lemma 3.2]). *Let A be a unital C^*-algebra that either has strict comparison or is purely infinite. Let $\beta : A \longrightarrow A$ be an injective endomorphism such that $\beta(1) \neq 1$ and $\beta(A)$ is a hereditary sub-C^*-algebra and let $A \times_\beta \mathbb{N} = C^*(A, V)$. If does not exist any proper ideal I of A such that $\beta(I) \subseteq I$, then for every full projection $p \in A$ there exist a partial isometry $u \in A$ and $m \in \mathbb{N}$ such that $(V^*)^m u^* p u V^m = (V^*)^m V^m = 1$.*

Proof. We claim that there exists $m \in \mathbb{N}$ such that $V^m(V^m)^* \lesssim p$. Observe that if A is purely infinite then p is a properly infinite full projection. So, we have that $VV^* \in \overline{ApA} = A$. Hence, $VV^* \lesssim p$, so that $m = 1$ holds.

Now suppose that A has strict comparison. Then $T(A)$ is non-empty and compact. So, given any $k \in \mathbb{N}$ we set

$$\alpha = \inf\{\tau(p) : \tau \in T(A)\} \qquad \text{and} \qquad \gamma_k = \sup\{\tau(V^k(V^*)^k) : \tau \in T(A)\}.$$

Observe that, since p is full, we have that $\alpha > 0$. Now, we claim that there exists $n \in \mathbb{N}$ such that $\gamma_n < 1$. Indeed, it is enough to prove that there exists $n \in \mathbb{N}$ such that $1 - V^n(V^*)^n$ is a full projection. Let us construct the ideal

$$I := \overline{\text{span}}\{x(V^l(V^*)^l - V^{l+1}(V^*)^{l+1})y : l \geq 0,\, x, y \in A\} \neq 0.$$

It is clear that $\beta(I) \subseteq I$. Therefore, by Lemma 3.1, there exists $n \in \mathbb{N}$ such that

$$(1-VV^*)+\cdots+\beta^{n-1}(1-VV^*) = (1-VV^*)+\cdots+(V^{n-1}(V^*)^{n-1}-V^n(V^*)^n) = 1-V^n(V^*)^n,$$

is a full projection. Therefore $\gamma_n < 1$. By the same argument as in the proof of [29, Lemma 3.2], we have that $\tau(V^{nl}(V^*)^{nl}) \leq \gamma_n^l$ for every $l \in \mathbb{N}$. Then, there exists $l \in \mathbb{N}$ such that $\tau(V^{nl}(V^*)^{nl}) \leq \gamma_n^l < \alpha \leq \tau(p)$. Since A has strict comparison, we have that $V^{nl}(V^*)^{nl} \lesssim p$. So, there exists a partial isometry $u \in A$ such that $u^*u = V^{nl}(V^*)^{nl}$ and $uu^* \leq p$. Therefore $(V^*)^{nl}u^*puV^{nl} = (V^*)^{nl}(V^{nl}(V^*)^{nl})V^{nl} = 1$, so we are done. □

Lemma 3.3. *Let A be a C^*-algebra of real rank zero, and let $\beta : A \to A$ be an extendible injective endomorphism with $\beta(A)$ being hereditary such that $\mathbb{T}(\beta) = \mathbb{T}$. Then, given any $a \in A^\sim$ and any B hereditary sub-C^*-algebra of A we have that*

$$\mathit{inf}\,\{\|pa\beta(p)\| : p \text{ is a non-zero projection of } B\} = 0.$$

Proof. Let $a \in A^+$ and let B be a hereditary sub-C^*-algebra of A. Given $\varepsilon > 0$, by Theorem 1.17 there exists $x \in B_+$ with $\|x\| = 1$ such that $\|xa\beta(x)\| < \varepsilon/2$. Given $\delta > 0$, let $f_\delta : [0,1] \longrightarrow [0,1]$ be such that $f(t) = 1$ for every $t \in [1-\delta/2, 1]$ and such that $|f_\delta(t) - t| < \delta$ for every $0 \leq t \leq 1$. Take $\delta > 0$ such that $\|f_\delta(x)a\beta(f_\delta(x))\| < \varepsilon$. Let $C = \{y \in B : f_\delta(x)y = yf_\delta(x) = y\} \neq 0$. Notice that C is a hereditary sub-C^*-algebra of B. Since C has real rank zero, there exists a non-zero projection $p \in C$, and by construction $pf_\delta(x) = f_\delta(x)p = p$. Therefore

$$\|pa\beta(p)\| = \|pf_\delta(x)a\beta(f_\delta(x)p)\| \leq \|f_\delta(x)a\beta(f_\delta(x))\| < \varepsilon.$$

□

Corollary 3.4. *Let A be a C^*-algebra of real rank zero, and let $\beta : A \longrightarrow A$ be an extendible injective endomorphism with $\beta(A)$ being hereditary such that $\mathbb{T}(\beta^n) = \mathbb{T}$ for every $n > 0$. Then, given any $\varepsilon > 0$, $a_1, \dots, a_k \in A^\sim$ and $n_1, \dots, n_k \in \mathbb{N}$ and a projection $p \in A$, there exists a projection $q \in pAp$ such that*

$$\|qa_i\beta^{n_i}(q)\| < \varepsilon \qquad \text{for every } i \in \{1, \dots, k\}.$$

A C^*-algebra A is said to be *weakly divisible* if given any projection $p \in A$, there exists a unital $*$-homomorphism $M_2 \oplus M_3 \longrightarrow pAp$ [31, Lemma 5.2]. Conditions for a non-type I real rank zero C^*-algebra being weakly divisible are given in [31, Theorem 5.8]. In particular,

every simple non-type I C^*-algebra of real rank zero is weakly divisible. Observe that, if A is weakly divisible or purely infinite, then the following statement holds:

(†) Given any $n \in \mathbb{N}$ and $p \in A$ there exists $p_1, \dots, p_n \in A$ non-zero pairwise orthogonal subprojections of p with $p \in \overline{Ap_iA}$ for all i

Proposition 3.5. *Let A be a unital C^*-algebra of real rank zero satisfying (†), let $\beta : A \longrightarrow A$ be an injective endomorphism such that $\beta(A)$ is a hereditary sub-C^*-algebra of A, and let $A \times_\beta \mathbb{N} = C^*(A, V)$. If does not exist any proper ideal I of A such that $\beta(I) \subseteq I$, then given any non-zero projection $p \in A$ there exist a full projection $q \in A$ and $c \in A \times_\beta \mathbb{N}$ such that $q = cpc^*$.*

Proof. By Lemma 3.1 there exists $n \in \mathbb{N}$ such that $p+\beta(p)+\cdots+\beta^n(p)$ is a full positive element of A. Since A satisfies (†) there exist non-zero orthogonal projections $p_0, \dots, p_n \in A$ such that $p_0+\cdots+p_n \leq p$ with $p \in \overline{Ap_iA}$ for all $i \in \{0, \dots, n\}$. Observe that $p+\beta(p)+\cdots+\beta^n(p)$ lies in the ideal generated by $q' := p_0+\beta(p_1)+\cdots+\beta^n(p_n)$, so q' is also a full positive element of A. Denote $p'_i := \beta^i(p_i)$ for every $i \in \{0, \dots, n\}$. Now we are going to use induction on n to construct a projection $q \in A$ such that $p'_0 + \cdots + p'_n \in \overline{AqA}$. The case $n = 0$ is clear. Now, suppose that there exists a projection q_{k-1} such that $p'_0 + \cdots + p'_{k-1} \in \overline{Aq_{k-1}A}$.

Using the Riesz decomposition of $V(A)$ [34] we have $p'_k \sim a_k \oplus b_k$ such that $a_k \lesssim q_{k-1}$ and $b_k \lesssim 1 - q_{k-1}$. Let v_k be the partial isometry such that $v_k^* v_k \leq p'_k$ and $v_k v_k^* \leq 1 - q_{k-1}$. If we define the projection $q_k := q_{k-1} + v_k v_k^*$, then we have that $p'_1 + \cdots + p'_k \in \overline{Aq_kA}$. Therefore the projection $q := q_n$ is full. If we define $c := p_0 + v_1 V^1 p_1 + \cdots + v_n V^n p_n$, then we have that

$$cpc^* = cc^* = p_0 + v_1\beta(p_1)v_1^* + \cdots + v_n\beta^n(p_n)v_n^* = q\,,$$

as desired. □

Theorem 3.6. *Let A be a unital C^*-algebra of real rank zero satisfying (†) that has strict comparison, let $\beta : A \longrightarrow A$ be an injective endomorphism such that $\beta(1) \neq 1$ and $\beta(A)$ is a hereditary sub-C^*-algebra of A. If $A \times_\beta \mathbb{N}$ is simple and $\beta(1)$ is a full projection of A, then $A \times_\beta \mathbb{N}$ is purely infinite simple C^*-algebra.*

Proof. It is enough to prove that given a positive element $x \in A \times_\beta \mathbb{N}$ there exist $a, b \in A \times_\beta \mathbb{N}$ such that $axb = 1$. Let $E : A \times_\beta \mathbb{N} \longrightarrow A$ be the canonical faithful conditional expectation. So, $0 \neq E(x) = c \in A_+$. Then, for $\|c\| > \varepsilon > 0$ we have that the hereditary sub-C^*-algebra $\overline{(c-\varepsilon)_+A(c-\varepsilon)_+} \subseteq c^{1/2}Ac^{1/2}$ has real rank zero. Hence, there exists a non-zero projection $p = c^{1/2}yc^{1/2} \in c^{1/2}Ac^{1/2}$. Then, $q = y^{1/2}cy^{1/2}$ is a projection, and $E(y^{1/2}xy^{1/2}) = y^{1/2}cy^{1/2} = q$. Thus, we can assume that $E(x) = q$ is a non-zero projection. Given $1/2 > \varepsilon > 0$, there exists $x' = (V^*)^m d_{-m} + \cdots + q + \cdots + d_m V^m$, with $d_j \in A_+$ for every j, such that $\|x - x'\| < \varepsilon$. By Corollary 1.18, Theorem 1.17 and Corollary 3.4, there exists a non-zero projection $p \in qAq$ such that

$$\|pd_i\beta^i(p)\| < \varepsilon/2m \qquad \text{and} \qquad \|\beta^i(p)d_{-i}p\| < \varepsilon/2m$$

for every $i \in \{1, \dots, m\}$. Therefore

$$\|pxp - p\| \leq \|pxp - px'p\| + \|px'p - p\| \leq \varepsilon + \varepsilon < 1\,.$$

Then, pxp is invertible in $p(A \times_\beta \mathbb{N})p$, whence there exists $y \in p(A \times_\beta \mathbb{N})p$ such that $ypxp = p$. Since we are assuming that $A \times_\beta \mathbb{N}$ is simple and $\beta(1)$ is a full projection, [32, Theorem 4.1]

implies that there are no non-trivial ideals I of A such that $\beta(I) \subseteq I$. Thus, by Proposition 3.5, there exist $c \in A \times_\beta \mathbb{N}$ and a full projection $q \in A$ such that $cpc^* = q$.

By Lemma 3.2, there exist $m \in \mathbb{N}$ and a partial isometry $u \in A$ such that $(V^*)^m u^* q u V^m = 1$ and therefore

$$(V^*)^m u^*(cypxpc^*)uV^m = (V^*)^m u^* cpc^* uV^m = (V^*)^m u^* quV^m = 1\,.$$

Thus, if we set $a := (V^*)^m u^* cyp$ and $b := pc^* uV^m$ we have $axb = 1$, as desired. □

When A is a purely infinite C^*-algebra, we generalize the result of [16].

Corollary 3.7. *Let A be a unital purely infinite C^*-algebra of real rank zero, let $\beta : A \longrightarrow A$ be an injective endomorphism such that $\beta(1) \neq 1$ is a full projection and $\beta(A)$ is a hereditary sub-C^*-algebra of A. Then $A \times_\beta \mathbb{N}$ is a simple purely infinite C^*-algebra if and only if $A \times_\beta \mathbb{N}$ is simple.*

Proof. The proof works in the same way as that of Theorem 3.6, but reminding that Lemma 3.2 and condition (†) are also satisfied for purely infinite C^*-algebras. □

Finally, we can use Corollary 3.7 to characterize when a crossed product by an automorphism $A \times_\alpha \mathbb{Z}$ is simple and purely infinite.

Corollary 3.8 (cf. [16, Theorem 3.1])**.** *Let A be a unital purely infinite C^*-algebra of real rank zero, and let $\alpha : A \longrightarrow A$ be an automorphism. Then $A \times_\alpha \mathbb{Z}$ is a simple purely infinite C^*-algebra if and only if $A \times_\alpha \mathbb{Z}$ is simple.*

Proof. The proof is a verbatim of the proof of [16, Theorem 3.1]. We only have to prove that there exist projections $p, e \in A$ and partial isometries $t, s \in A$ such that

$$s^*s = \alpha(p)\,, \qquad ss^* = e < p\,, \qquad t^*t = 1 - \alpha(p) \qquad \text{and} \qquad tt^* = 1 - e\,.$$

Indeed, since 1 is a properly infinite projection, there exist mutually orthogonal projections $p_1, p_2, p_3 \in A$, all them Murray-von Neumann equivalent to 1. Observe that $\alpha(p_i)$ are mutually orthogonal full properly infinite projection of A. Then, we have that $\alpha(p_1) \sim e < \alpha(p_2)$ for some projection $e \in A$. Since $\alpha(p_3) \perp \alpha(p_1)$ and $e \perp \alpha(p_1)$, by [10, Proposition 2.5] we have that $\alpha(p_1)$ and e are homotopic equivalent, and hence $1 - \alpha(p_1)$ and $1 - e$ are homotopic equivalent, thus Murray-von Neumann too. Thus, setting $p := \alpha(p_1)$ we have proved the claim.

By the proof of [16, Theorem 3.1], the dynamical system (A, α) is exterior equivalent to (A, ρ), where ρ is the automorphism defined by $\rho(x) = (s+t)\alpha(x)(s+t)^*$ for every $x \in A$. So, it is enough to prove that $A \times_\rho \mathbb{Z}$ is simple and purely infinite. Notice that $\mathbb{T}(\alpha) = \mathbb{T}(\rho)$, and that A is ρ-simple since it is α-simple. Hence, $A \times_\rho \mathbb{Z}$ is a simple C^*-algebra. Then $p(A \times_\rho \mathbb{Z})p \cong pAp \times_\rho \mathbb{N}$ is a full simple hereditary sub-C^*-algebra. Now, we have that pAp is a purely infinite C^*-algebra of real rank zero, and by construction $\rho(p) = s\alpha(p)s^*$ is a full projection of pAp. Thus, by Theorem 3.7 we have that $pAp \times_\rho \mathbb{N}$ is a purely infinite C^*-algebra, whence so is $A \times_\alpha \mathbb{Z}$. □

Example 3.9. This is a generalization of Example 2.5(3) and Cuntz's construction of the algebras $\mathcal{O}_n$ [12]. Let $\mathcal{U}_m$ be the m-infinity UHF algebra $\bigotimes_{n=1}^{\infty} M_m$, and let $B = \mathcal{U}_m \oplus \cdots \oplus \mathcal{U}_m$ be the direct sum of n copies of $\mathcal{U}_m$, that is a nuclear unital weakly divisible C^*-algebra of real rank zero that absorbs $\mathcal{Z}$ and hence has strict comparison. Let us consider the endomorphism

$\beta : B \longrightarrow B$ given by $\beta(x_1, \dots, x_n) = (P_1 \otimes x_2, P_2 \otimes x_3 \cdots, P_n \otimes x_1)$ for every $(x_1, \dots, x_n) \in B$, where $P_1, \cdots, P_n \in M_m$ are rank 1 projections. Hence, β is injective. Observe that $\beta(1) \neq 1$ is a full projection of B. It is clear that B is β-simple and β^k is outer for any $k > 0$, since B is a unital finite C^*-algebra. Hence, $B \times_\beta \mathbb{N}$ is simple by Theorem 1.17, and thus applying Theorem **??** it is also a purely infinite C^*-algebra, in particular it is a Kirchberg algebra. Now, we use the modification of the Pimsner-Voiculescu six-term exact sequence given in [29],

$$\begin{tikzcd} K_0(B) \arrow[r, "1-\beta^*"] & K_0(B) \arrow[r] & K_0(B \times_\beta \mathbb{N}) \arrow[d] \\ K_1(B \times_\beta \mathbb{N}) \arrow[u] & K_1(B) \arrow[l] & K_1(B) \arrow[l, "1-\beta^*"'] \end{tikzcd}$$

Notice that the induced map $\beta^* : \mathbb{Z}[1/m]^n \longrightarrow \mathbb{Z}[1/m]^n$ is given by

$$\beta^*(x_1, \dots, x_n) = (x_2/m, \dots, x_n/m, x_1/m)\,,$$

for every $(x_1, \dots, x_n) \in \mathbb{Z}[1/m]^n$. Then, we can easily compute $K_0(B \times_\beta \mathbb{N}) = \mathbb{Z}/(m^n - 1)\mathbb{Z}$ and $K_1(B \times_\beta \mathbb{N}) = 0$. Thus, using the Kirchberg-Phillips classification theorems, we conclude that $B \times_\beta \mathbb{N}$ is stably isomorphic to the Cuntz algebra $\mathcal{O}_{m^n}$.

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Department of Mathematical Sciences, NTNU, NO-7491 Trondheim, Norway
E-mail address: eduardor@math.ntnu.no

Departamento de Matemáticas, Facultad de Ciencias, Universidad de Cádiz, Campus de Puerto Real, 11510 Puerto Real (Cádiz), Spain.
E-mail address: enrique.pardo@uca.es
URL: https://sites.google.com/a/gm.uca.es/enrique-pardo-s-home-page/